On the Multiplication of Semi-convergent Series.

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In vol. 24, p. 44 of Math. Ann., A. Voss has given the necessary and sufficient conditions which must be satisfied in order that Cauchy's rule for the multiplication of series be applicable to semi-convergent series Σa_n and Σb_n, in case that one of them, say Σa_n, becomes absolutely convergent when expressed in the form $\Sigma(a_{2n} + a_{2n+1})$. The purpose of the present article is to extend Voss's results.

It is always possible to find a series converging toward the product of the sums of two semi-convergent series, when one of the factor-series can be made absolutely convergent on associating its terms into groups, each containing a finite number of terms. Thus, if $U_n = \sum_0^n a_n$ and $V_n = \sum_0^n b_n$ are semi-convergent, and if $U_n = \sum_0^n g_n$ is absolutely convergent (g_n being the $(n+1)^{\text{th}}$ group in the first series), then by a theorem of Mertens,

$$UV = \sum_0^\infty (b_n g_0 + b_{n-1} g_1 + \ldots + b_0 g_n).$$

If the product $\sum_0^n (a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0)$, of $\sum_0^n a_n$ and $\sum_0^n b_n$, formed according to Cauchy's rule, is convergent, then by a theorem of Abel, it converges to UV. From this it follows that the necessary and sufficient condition for the convergence of the product-series is that

$$\sum_0^\infty (b_n g_0 + b_{n-1} g_1 + \ldots + b_0 g_n) = \sum_0^\infty (a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0') \qquad \text{(I)}$$

Let us suppose that all the groups contain the same number p of terms, so that $g_n = a_{pn} + a_{pn+1} + \dots + a_{pn+p-1}$, then for the case $n = pm$,

$$\sum_0^n (b_n g_0 + b_{n-1} g_1 + \dots + b_0 g_n) - \sum_0^n (a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0)$$
$$= b_0 \{a_{pm+1} + a_{pm+2} + \dots + a_{p^2m+p-1}\}$$
$$+ b_1 \{a_{pm} + a_{pm+1} + \dots + a_{p^2m-1}\} + \dots + b_{pm} \{a_1 + a_2 + \dots + a_{p-1}\}$$
$$= b_0 (a_{pm+1} + \dots + a_{pm+p-1})$$
$$+ \sum_{i=0}^{i=m-1} \{b_{pi+2} a_{pm-pi-1} + b_{pi+3} (a_{pm-pi-2} + a_{pm-pi-1}) + \dots$$
$$+ b_{pi+p} (a_{pm-pi-p+1} + a_{pm-pi-p+2} + \dots + a_{pm-pi-1})\} + E.$$

If $pm = 2ps$, then

$$\begin{aligned} E \equiv\, & g_1 \{b_{2ps-1} + b_{2ps-2} + \dots + b_{2ps-p+1}\} \\ & + g_2 \{b_{2ps-2} + b_{2ps-3} + \dots + b_{2ps-2p+1}\} + \dots \\ & + g_s \{b_{2ps-s} + b_{2ps-s-1} + \dots + b_{ps+1}\} \\ & + g_{s+1}\{b_{2ps-s-1} + b_{2ps-s-2} + \dots + b_{ps-p+1}\} + \dots + g_{2ps-1}\{b_1 + b_0\} + g_{2ps} b_0. \end{aligned}$$

If $pm = 2ps + 1$, then

$$\begin{aligned} E \equiv\, & g_1 \{b_{2ps} + b_{2ps-1} + \dots + b_{2ps-p+2}\} \\ & + g_2 \{b_{2ps-1} + b_{2ps-2} + \dots + b_{2ps-2p+2}\} + \dots \\ & + g_s \{b_{2ps-s+1} + b_{2ps-s} + \dots + b_{ps+2}\} \\ & + g_{s+1} \{b_{2ps-s} + b_{2ps-s-1} + \dots + b_{ps-p+2}\} + \dots + g_{2ps}\{b_1 + b_0\} + g_{2ps+1} b_0. \end{aligned}$$

In either case a quantity β and an infinitesimal quantity, ϵ_s, approaching zero as s increases indefinitely, can be so chosen that, for large values of s (letting $|x|$ stand for the absolute value of x),

$$|E| < \epsilon_s \{|g_1| + |g_2| + \dots + |g_s|\} + \beta \{|g_{s+1}| + |g_{s+2}| + \dots + |g_{pm}|\}.$$

Since Σg_n is absolutely convergent, it follows that the second member of the inequality approaches the limit zero as s increases indefinitely. Therefore E approaches zero as a limit, and the condition that equation (I) be satisfied for $n = pm$, is

$$\lim_{m=\infty} \sum_{i=0}^{i=m-1} \{b_{pi+2} a_{pm-pi-1} + b_{pi+3} (a_{pm-pi-2} + a_{pm-pi-1}) + \dots$$
$$+ b_{pi+p} (a_{pm-pi-p+1} + a_{pm-pi-p+2} + \dots + a_{pm-pi-1})\} = 0.$$

By a similar process of reasoning we obtain the condition that (I) be satisfied, for the case $n = pm + r$, viz.

$$\underset{m=\infty}{Lim} \sum_{i=0}^{i=m-1} \{ b_{pi+r+2} a_{pm-pi-1} + b_{pi+r+3} (a_{pm-pi-2} + a_{pm-pi-1}) + \dots \\ + b_{pi+p+r} (a_{pm-pi-p+1} + \dots + a_{pm-pi-1}) \} = 0 . \quad \text{(II)}$$

If we agree to let r represent successively all integral values from 0 to $p - 1$ (both inclusive), then *expression* (II) *embodies* p *equations which constitute together the necessary and sufficient conditions for the existence of equation* (I) *and for the applicability of Cauchy's multiplication rule to* Σa_n and Σb_n.

Another set of necessary and sufficient conditions can be deduced from conditions (II), viz.

Cauchy's multiplication rule is applicable to Σa_n *and* Σb_n, *if the* n^{th} *term of the product-series always approaches the limit zero and if* ONE *of the* p *conditions in* (II) *is satisfied.*

We first prove that if the n^{th} term of the product-series approaches zero as n increases indefinitely and if the $(r + 1)^{\text{th}}$ condition in (II) is satisfied, then the r^{th} condition in (II) is also satisfied. We have (disregarding, as we may, a finite number of terms $a_{n-x} b_x$ in which $x < r + 2$ or $x > pm + r$),

$$\underset{m=\infty}{Lim} \Bigg[\sum_{i=0}^{i=m-1} \{ b_{pi+r+2} a_{pm-ip-1} + b_{pi+r+3} (a_{pm-pi-1} + a_{pm-pi-2}) + \dots \\ + b_{pi+p+r} (a_{pm-pi-1} + \dots + a_{pm-ip-p+1}) \} \\ + \sum_{i=0}^{i=m} g_{m-i} (b_{pi+r+1} + b_{pi+r+2} + \dots + b_{pi+p+r-1}) \\ - \sum_{r=r}^{r=p+r-2} (a_{pm+r+1} b_0 + a_{pm+r} b_1 + \dots + a_0 b_{pm+r+1}) \Bigg]$$

$$= \underset{m=\infty}{Lim} \sum_{i=0}^{i=m} \{ b_{pi+r+1} a_{pm+p-pi-1} + b_{pi+r+2} (a_{pm+p-pi-1} + a_{pm+p-pi-2}) + \dots \\ + b_{pi+p+r-1} (a_{pm+p-ip-1} + \dots + a_{pm-ip+1}) \} .$$

In the first member of this equation we assume the $(r + 1)^{\text{th}}$ condition of (II) to be satisfied and the sum of $p - 1$ successive terms in the product-series to approach zero. Remembering that Σg_n is absolutely convergent, it will be seen that the first member of the equation approaches the limit zero; hence the second member approaches zero. If in the second member we put $m-1$ in place of m, then the expression assumes the form of the r^{th} condition in (II). Hence if the $(r+1)^{\text{th}}$ condition is satisfied, the r^{th} condition is satisfied. But if the r^{th} condition is satisfied, then the $(r-1)^{\text{th}}$ is satisfied, etc., and the theorem is established.

A set of *sufficient* conditions of convergence is obtained by taking the absolute values of all the a's and all the b's in (II). In the light of the conditions thus obtained one may readily see the correctness of a condition established by A. Pringsheim (Math. Annalen, Vol. 21, p. 334), which asserts that the product-series is convergent and Cauchy's rule applicable, if $\sum_0^n a_n b_n$ is absolutely convergent and remains so when any number of factors a_m, b_m is replaced by other factors of higher indices. Pringsheim proves that when his condition is satisfied, then the n^{th} term of the product-series approaches zero. Moreover, if $\sum_0^n a_n b_n$ converges absolutely, then not only is $\sum_n^\infty a_n b_n = 0$, but zero is also the limit of the sum of all the terms $a_x b_y$ in the expression, after making substitutions of the kind above referred to, provided that $x + y \gtreqless 2n$. But the values of the indices x and y may be so chosen that the absolute values of all the products ab involved in any one of the p conditions in expression (II) will be included. Hence Pringsheim's *sufficient* condition is true, for whenever that condition is satisfied, then the *necessary* and *sufficient* conditions developed in this paper are satisfied. But observe that we have verified Pringsheim's condition only for the case that Σa_n becomes absolutely convergent on associating its terms into groups containing an equal number p of terms, while his condition is also applicable to cases in which the number of terms in the various groups is not the same.

As an example, consider the product of the two semi-convergent series

$$\sum_1^m\left(\frac{1}{3m}+\frac{1}{3m+1}-\frac{2}{3m+2}\right) \text{ and } \sum_1^m\left\{\frac{(-1)^m}{\log 3m}+\frac{(-1)^m}{\log(3m+1)}+\frac{(-1)^m}{\log 3m+2)}\right\}$$

The first of these becomes absolutely convergent when $p=3$. Pringsheim's test fails here, for $\sum_0^n a_n b_n$ is not absolutely convergent. Applying the second test developed in this paper, we observe that the n^{th} term of the product-series approaches the limit zero as n increases indefinitely, no matter whether n equals $3m$, $3m+1$ or $3m+2$. The first condition in (II) is also satisfied, for

$$\operatorname*{Lim}_{m=\infty}\Big[\Big\{+\frac{1}{\log 5}\cdot\frac{2}{3m-4}-\frac{1}{\log 8}\cdot\frac{2}{3m-7}+\frac{1}{\log 11}\cdot\frac{2}{3m-10} - \ldots \pm\frac{1}{\log(3m-4)}\cdot\frac{2}{5}\Big\}$$
$$+\Big\{-\frac{1}{\log 3}\Big(\frac{1}{3m-2}-\frac{2}{3m-1}\Big)+\frac{1}{\log 6}\Big(\frac{1}{3m-5}-\frac{2}{3m-4}\Big) - \ldots \pm\frac{1}{\log(3m-3)}\Big(\frac{1}{4}-\frac{2}{5}\Big)\Big\}\Big]=0.$$

Hence the product-series converges and Cauchy's multiplication rule is applicable.

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